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Back Analysis of Measured Displacements of Tunnels

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Summary

The present paper deals with a method of back analysis to be utilized for the interpretation of field measurements in monitoring the stability of tunnels. The method belongs to an inverse approach based on the finite element formulation, assuming the ground media in which tunnels are excavated to be linear, isotropic and elastic. Assuming Poisson's ratio and the vertical initial stress, the method derives the complete initial state of stress and Young's modulus from a set of relative displacements measured between adjacent measuring points. In order to verify the applicability of the proposed method of back analysis to practical engineering problems, a case study is presented.

1. Introduction

The knowledge of the initial state of stress in the ground and of the material properties is of fundamental importance both for understanding the structural behaviour and modelling the reality numerically. The initial stresses, as well as the material constants, may be determined by various types of in-situ tests. The stress relief method by overcoring is the most popular for determining the initial stresses. Plate-bearing tests and direct shear tests are extensively used, particularly in dam construction sites, for determining the deformability and strength of rock masses. It is well known, however, that the results of these tests in general indicate a certain amount of scatter because of non-homogeneous characteristics of the ground, though the tests are conducted at nearly the same geological formations. Besides this, the mechanical properties of the ground must be evaluated in relation to the size and shape of the structures. This means that the values obtained from the in-situ tests cannot be used directly as input data for an analysis, and so great care should be taken when interpreting the results of the tests.

In order to overcome these difficulties, field observation of the deformational behaviour of tunnels during construction seems to be most promising. Observations by means of displacement measurements such as tunnel convergence and borehole extensometer measurements are desirable because

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of the simplicity of instrumentation and reliability of the data obtained, while direct stress measurements are always less reliable. Such displacement measurements may then be used to back-calculate the initial stresses and the material properties.

Back analysis problems may be solved in two different ways defined as inverse and direct approaches (Cividini et al., 1981). In the *inverse approach* the formulation is just the reverse of that in the ordinary stress analysis, although the governing equations are identical. However, the number of the measured values should be greater than the number of unknown parameters, so that optimization techniques can be used to determine the unknowns. For simple geometrical configurations and material models, closed form solutions may be found in the literature. For structures with arbitrary shape under more complex geological conditions the Finite Element Method (FEM) seems to be most promising. For example Kavanagh (1973) proposed a back analysis formulation based on the FEM which may make it possible to get the material constants, not only for isotropic materials, but also for inhomogeneous and anisotropic materials from both measured displacements and strains. However, it is likely that if the input data of displacements or strains indicate a certain scatter, Kavanagh's method never converges. Therefore it is questionable if the method may be applied to geotechnical engineering problems in which the measured values always contain a scattering. The *direct approach* to back analysis is based on an iterative procedure correcting the trial values of unknown parameters by minimizing error functions. The advantage is that this method may be applied to non-linear back analysis problems without having to rely on a complex mathematical background (Gioda and Maier, 1980) and, in addition, that standard algorithms of mathematical programming may be adopted for the numerical solution (Cividini et al., 1981). On the other hand, the iterative solution requires quite time-consuming computations.

In this paper an inverse approach to back analysis is proposed, which is based on a finite element formulation.

2. Objectives and Assumptions

For analysing the stability of tunnels, Sakurai (1981, 1982) proposed the Direct Strain Evaluation Technique, which allows a quantitative interpretation of the results of displacement measurements. The technique is based on the concept of strain rather than stress. Since strain is kinematically related to displacement, strain distributions in the ground can be obtained directly from the measured displacements without performing any stress analyses. The stability of tunnels is then assessed by comparing the derived strains with critical failure strains of the soil or rock. The main objective of the back analysis technique outlined here is to determine the initial stresses and material constants from measured displacements. This data may then be used as input data to compute the strain distribution in the vicinity of a tunnel.

The following assumptions are made:

— The mechanical behaviour of the ground is idealized by an isotropic linear-elastic model, so that the material constants reduce to Young's modulus and Poisson's ratio. Since Poisson's ratio has less influence on the results of an analysis, an appropriate value will be assumed.
— The elastic constants of the lining are assumed to be known.
— The initial state of stress is assumed to be constant all over the region under consideration.

3. Formulation of Back Analysis

Genuine rock pressure may be taken into account in an FE-computation by applying equivalent nodal forces $\{P^0\}$ at the excavation surface, which correspond to the initial state of stress at this place. These forces may be determined by

$$\{P^0\} = \int_V [B]^T \{\sigma^0\}\, dV \tag{1}$$

where $[B]$: matrix depending on geometry only,
$\{\sigma^0\}$: initial stresses,
V: excavation volume.

For the sake of simplicity, in this study a two-dimensional formulation is presented, whereby one principal axis (z-axis) is orientated parallel to the tunnel axis. With the components of the initial stress being defined in a Cartesian coordinate system x, y, z by

$$\{\sigma^0\} = \{\sigma_x{}^0\ \sigma_y{}^0\ \tau_{xy}{}^0\ \sigma_z{}^0\}^T$$

the vector $\{P^0\}$ may be expressed according to

$$\{P^0\} = \sigma_x{}^0 \{B^1\} + \sigma_y{}^0 \{B^2\} + \tau_{xy}{}^0 \{B^3\} \tag{2}$$

where the vectors

$$\{B^i\} = \{B_1{}^i\ B_2{}^i\}^T \qquad (i = 1, 2, 3)$$

are obtained from (1) by integration over the excavation volume, i. e. $\{B^i\}$ depend on geometry only. The relation between the nodal forces $\{P\}$ of the system and the nodal displacements $\{u\}$ may be expressed by the well known relationship

$$\{P\} = [K]\{u\} \tag{3}$$

where $[K]$ denotes the stiffness matrix of the assembled finite element system. For a lined tunnel with the Young's moduli E^R and E^L of the rock and the lining respectively the stiffness matrix may be split into

$$[K] = E^R \left([K^R] + \frac{E^L}{E^R}[K^L]\right) = E^R [K^*] \tag{4}$$

where $[K^R]$ represents the stiffness matrix for the ground for $E^R=1$ and $[K^L]$ that for the lining for $E^L=1$. If the FE-mesh is chosen in such a way that the measuring points coincide with the nodes of the mesh, the displacements may be grouped in a vector $\{u^m\}$ which contains measured points only and a vector $\{u^u\}$ which represents the unknown displacements at the rest of the nodal points

$$\{u\} = \{u^m\} + \{u^u\}.$$

If this relation together with the stiffness matrix $[K^*]$ expressed by (4) is substituted in (3) for the load vector $\{P\}=\{P^0\}$ according to (2) one obtains

$$\sigma_x{}^0 \begin{Bmatrix} B_1{}^1 \\ B_2{}^1 \end{Bmatrix} + \sigma_y{}^0 \begin{Bmatrix} B_1{}^2 \\ B_2{}^2 \end{Bmatrix} + \tau_{xy}{}^0 \begin{Bmatrix} B_1{}^3 \\ B_2{}^3 \end{Bmatrix} = E^R \begin{bmatrix} K_{11}{}^* & K_{12}{}^* \\ K_{21}{}^* & K_{22}{}^* \end{bmatrix} \begin{Bmatrix} u^m \\ u^u \end{Bmatrix}.$$

Solving for $\{u^u\}$ and substitution leads to

$$\sigma_x{}^0 \{B_x\} + \sigma_y{}^0 \{B_y\} + \tau_{xy}{}^0 \{B_{xy}\} = E^R [K_N{}^*] \{u^m\}$$

with

$$\{B_x\} = \{B_1{}^1\} - [K_{12}{}^*] [K_{22}{}^*]^{-1} \{B_2{}^1\},$$
$$\{B_y\} = \{B_1{}^2\} - [K_{12}{}^*] [K_{22}{}^*]^{-1} \{B_2{}^2\},$$
$$\{B_{xy}\} = \{B_1{}^3\} - [K_{12}{}^*] [K_{22}{}^*]^{-1} \{B_2{}^3\},$$
$$[K_N{}^*] = [K_{11}{}^*] - [K_{12}{}^*] [K_{22}{}^*]^{-1} [K_{21}{}^*].$$

This equation may be rewritten as

$$\{u^m\} = \frac{1}{E^R} [A] \{\sigma^0\} = [A] \{\bar{\sigma}^0\} \tag{5}$$

where the matrix $[A]$

$$[A] = [[K_N{}^*]^{-1} \{B_x\} \; [K_N{}^*]^{-1} \{B_y\} \; [K_N{}^*]^{-1} \{B_{xy}\}]$$

depends on geometry and the Poisson's ratios for both the ground and the lining. The ratio $\{\bar{\sigma}_0\}$ of the initial stress to Young's modulus of the rock is called normalized initial stress. Thus (5) contains three unknown values

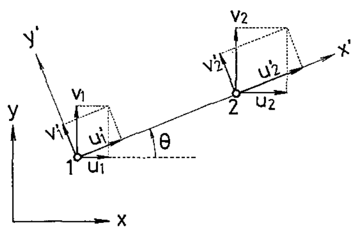

Fig. 1. Displacement components at two adjacent measuring points

of normalized initial stress and consists of as many equations as measured displacement components. If the number of measured displacements is greater than three, the normalized initial stresses can be determined with an optimization procedure.

It should be noted, however, that in practice it is easier to measure relative displacements between two different measuring points than absolute displacements, so that $\{u^m\}$ in Eq. (5) should be transformed to relative displacements. The relative displacements between adjacent measuring points shown in Fig. 1 can be expressed in terms of absolute displacements as follows:

$$\{\varDelta u\} = \begin{Bmatrix} u_2' - u_1' \\ v_2' - v_1' \end{Bmatrix} = \begin{bmatrix} -\cos\Theta & -\sin\Theta & \cos\Theta & \sin\Theta \\ \sin\Theta & -\cos\Theta & -\sin\Theta & \cos\Theta \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}. \tag{6}$$

Hence the absolute displacements $\{u^m\}$ are related to the measured relative displacements by

$$\{\varDelta u^m\} = [T]\,\{u^m\}. \tag{7}$$

The transformation matrix $[T]$ is obtained with the consideration of (6). Substituting (5) into (7) yields

$$\{\varDelta u^m\} = [A^*]\,\{\bar{\sigma}^0\} \tag{8}$$

with

$$[A^*] = [T]\,[A].$$

If the least squares method is adopted in (8) the normalized initial stresses can be determined uniquely from the measured relative displacements by

$$\{\bar{\sigma}^0\} = [[A^*]^T\,[A^*]]^{-1}\,[A^*]^T\,\{\varDelta u^m\}. \tag{9}$$

4. Calculation Procedure

4.1 Unlined Tunnels

Since the ratio E^L/E^R is zero for unlined tunnels, the matrix $[A^*]$ in (9) is defined uniquely for a given Poisson's ratio ν^R of the rock. Assuming ν^R and substitution of $\{\varDelta u^m\}$ by a set of measured relative displacements leads therefore immediately to the normalized initial stress. If the value of the initial stress itself is required, a relation between this stress and Young's modulus E^R would be necessary. Because such a relation does not exist one component of the initial stress or E^R must be determined in some other way. We propose to assume the vertical component $\sigma_y{}^0$ proportional to the overburden H and the specific weight γ^R of the ground

$$\sigma_y{}^0 = \gamma^R\,H. \tag{10}$$

Hence E^R is determined by

$$E^R = \frac{\gamma^R H}{\bar{\sigma}_y{}^0} \tag{11}$$

and the other two components of the initial stress by

$$\sigma_x{}^0 = E^R\,\bar{\sigma}_x{}^0,$$
$$\tau_{xy}{}^0 = E^R\,\bar{\tau}_{xy}{}^0.$$

It should be noted, however, that if only the evaluation of the strain distribution is required, there is no need to determine the initial stresses and Young's modulus separately. Instead, only the ratio of these quantities is important.

4.2 *Lined Tunnels*

In the case of lined tunnels, the matrix $[A^*]$ is not known a priori even when the Poisson's ratio ν^L of the lining and E^L are assumed, because it depends on the ratio E^L/E^R. Therefore an iterative solution must be adopted. For this an adequate value of E^R is assumed, the procedure described for unlined tunnels is carried out and the resulting value for E^R defined by (11) is compared with the assumed one. If the two values differ more than a prescribed limit, this procedure is repeated with an adjusted value of E^R until satisfying the limit.

5. Illustrative Example

An example from engineering practice demonstrates the applicability of the proposed method for practical problems. For a tunnel with an overburden of approximately 400 m convergence and extensometer measurements were started right after excavation. The arrangement of the measurements together with the readings at the final stage of construction are represented in Fig. 2.

Disregarding the effects of the lining the back analysis was carried out as outlined in ch. 4.1. Assumption of $\nu^R = 0.3$ led immediately to the normalized initial stresses

$$\{\bar{\sigma}^0\} = \begin{Bmatrix} \bar{\sigma}_x{}^0 \\ \bar{\sigma}_y{}^0 \\ \bar{\tau}_{xy}{}^0 \end{Bmatrix} = \begin{Bmatrix} 0.0489 \\ 0.0609 \\ 0.0003 \end{Bmatrix}.$$

Using in a second analysis the data of the convergence measurements only, the results were

$$\{\bar{\sigma}^0\} = \begin{Bmatrix} 0.0490 \\ 0.0587 \\ -0.0001 \end{Bmatrix}.$$

Comparing these values it may be seen that the difference between the two sets is very small. This means that for this case the convergence measurements alone would be sufficient for a reliable back analysis.

The normalized initial stresses obtained may now be used to determine the initial stresses. Assumption of the vertical component

$$\sigma_y{}^0 = 0.0245 \cdot 400 = 9.80 \text{ MPa}$$

according to (10) leads to

$$\begin{aligned} E^R &= 160.9 \text{ MPa}, \\ \sigma_x{}^0 &= 7.87 \text{ MPa}, \\ \tau_{xy}{}^0 &= 0{,}04 \text{ MPa}. \end{aligned}$$

The displacement and strain distributions in the vicinity of the tunnel are now computed by means of an ordinary finite element analysis utilizing as input data the initial stresses and Young's modulus obtained here. The corresponding results are illustrated in Fig. 2. It may be seen that there is a fairly good agreement between the computed and measured displacements.

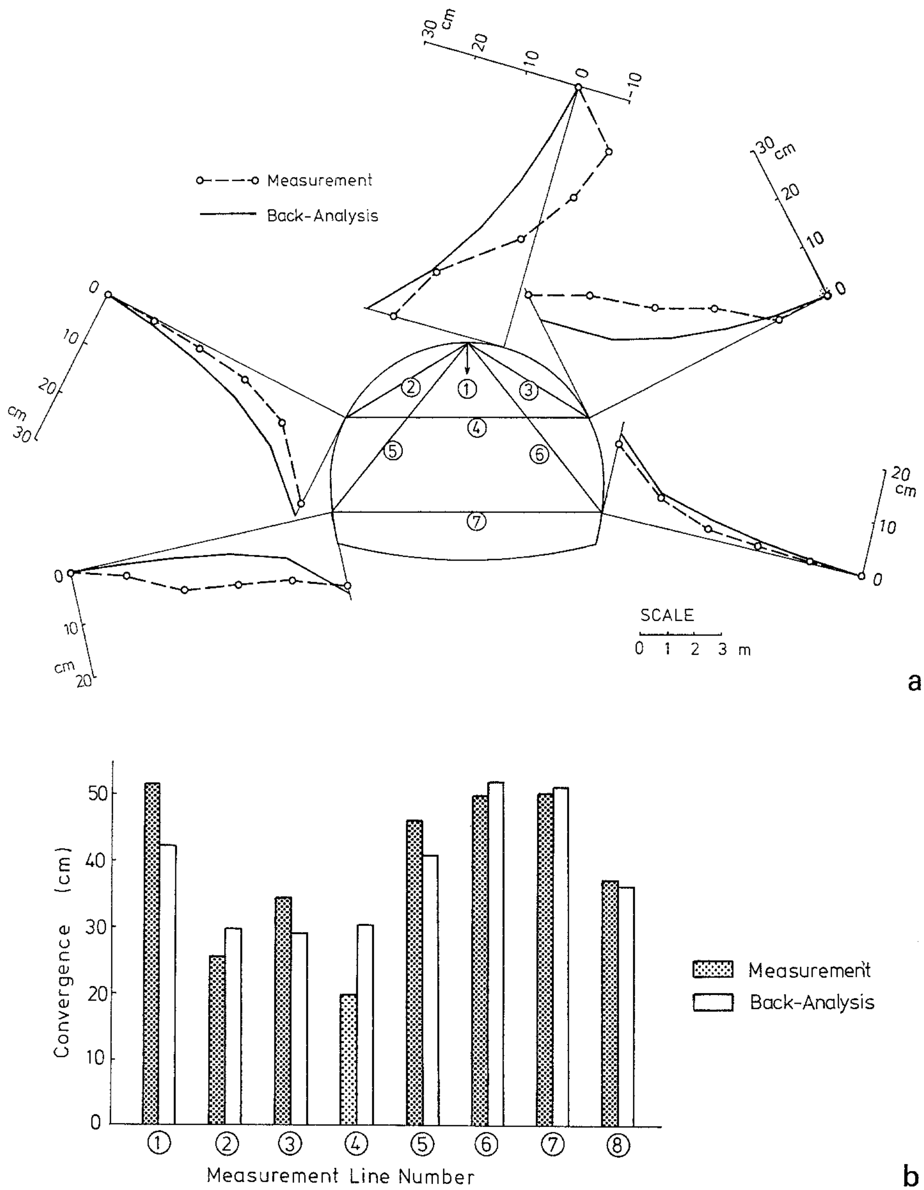

Fig. 2. Comparison between the measured and back-analyzed displacements
a) Extensometer measurements; b) Convergence measurements

6. Conclusions

In this paper a back analysis procedure based on the FEM has been proposed for determining the initial stresses $\{\sigma^0\}$ and Young's modulus E^R of the rock mass from displacement measurements. The main conclusions may be summarized as follows:

— If only the normalized stresses $\frac{1}{E^R}\{\sigma^0\}$ are required, assumption of the Poisson's ratio ν^R leads to the desired results. If E^R and $\{\sigma^0\}$ should be determined separately, one component of $\{\sigma^0\}$ or E^R must be assumed.

— In the proposed method relative displacements between two measuring points can be taken into account. This is an advantage because in practice measuring relative displacements is much easier than absolute ones.

— In the case of lined tunnels a small number of iterations is necessary for the calculation of the initial stresses, while in unlined tunnels a single calculation is good enough.

— Compared with similar methods the proposed procedure is numerically very stable. Thus it may be applied also for measured data with a large scatter.

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