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THEME/FEATURE PAPER

BACK ANALYSIS PROCEDURES FOR THE INTERPRETATION OF FIELD MEASUREMENTS IN GEOMECHANICS

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SUMMARY

A survey is presented of some recent developments of the numerical techniques for back analysis in the field of geomechanics, with particular reference to tunnelling problems. In the spirit of Terzaghi's observational design method, these techniques are seen as practical tools for interpreting the available field measurements, in order to reduce the uncertainties that in many instances affect the parameters governing the solution of complex geomechanics problems. Both deterministic and probabilistic viewpoints are considered and some significant applications to practical problems are illustrated.

INTRODUCTION

The design of the majority of rock (or soil) engineering works is customarily carried out in one of two ways, which have marked conceptual and operative differences. Following a first possible procedure, the experimental investigation aimed at defining the mechanical properties of the *in situ* rock mass is limited in time to the period preceding the beginning of the design process and, consequently, of the construction works. It is implicitly assumed that such an investigation provides meaningful values for the mechanical parameters of the rock mass and that no significant deviations from them will be encountered subsequently. As a consequence, even though the last assumption is often unrealistic, no *in situ* measurements are planned during construction and the possibility of modifying the final design, due to unforeseen conditions met in the field, is not accounted for. In this sense a 'stiff' design is produced.

Such a procedure, which does not involve any interaction between the geotechnical consultant and the contractor, presents non-negligible drawbacks.[1] In fact, in order to overcome the uncertainties caused by the limited experimental information available and to reduce the possibility of overestimating the rock mass quality, the designer might (a) adopt a large factor of safety, or (b) introduce hypotheses or pieces of information derived from cases similar to the one under examination.

While the first choice is evidently non-economical, the second one can be unsafe. In fact, geological details which may be easily overlooked during the preliminary investigation can significantly affect the overall behaviour of the rock mass and, as a consequence, apparently

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similar field conditions can actually lead to quite different construction problems requiring different design strategies.

In order to avoid the above drawbacks, an alternative procedure was introduced by Terzaghi for the design of dams and large geotechnical works.[2] This procedure is known as the observational design method and requires close co-operation between contractor and geotechnical consultant. A measurement program is set up, the characteristics of which depend on the problem at hand and which should not interfere appreciably with the field operations, so that the overall behaviour of the rock mass can be monitored during construction. Then, the experimental measurements (e.g. displacements) are checked against the corresponding quantities calculated on the basis of the rock mechanical parameters adopted in the design. If a non-negligible discrepancy is observed between the two sets of data, the rock mass parameters are modified in such a way that this discrepancy is reduced to a minimum. This leads to a new estimate of the rock mass properties which permits more refined calculations and possible modifications of the original design.

It can easily be recognized that in order to apply this second 'flexible' design procedure it is necessary to define a way for refining the values of the rock mass properties on the basis of the available *in situ* measurements. This is customarily referred to as the back analysis or characterization problem and can be stated as follows: find the values of the mechanical parameters that when introduced in the stress analysis of the problem under examination lead to results (e.g. displacements) as close as possible to the corresponding *in situ* measurements.

It is known that the experience and the judgement of the engineer are fundamental ingredients for every back analysis. They enable choosing, for instance, a suitably simplified geometrical model for a complex geological formation, or a reasonable material behaviour (elastic, elasto-plastic, viscous, etc.) for a specific rock type. Poor, or even incorrect, results of back analyses are often due to errors in these initial engineering choices.[3]

It is important to note, however, that the designer is not restricted to using his judgement only for solving the complex problem of discrepancy minimization between a set of field measurements and the results of stress analyses. In fact, numerical tools are available nowadays which can provide him with valuable help at an acceptable computational cost.

Techniques of this type, developed in the field of system identification,[4] have been successfully applied to various engineering problems, concerning structural dynamics (see, for example, references 5–7), soil dynamics (see, for example, reference 8) and structural plasticity (see, for example, References 9–11). Here a discussion is presented on some recent developments of the numerical techniques for back analysis and on their application in the field of rock (and soil) engineering,[12,13] with particular reference to tunnelling problems.

Before going into details it is necessary to point out that these procedures are applicable to a range of problems broader than that previously mentioned, concerning only the calibration of material models. For instance, techniques belonging to this category have been developed for determining the earth pressure distribution acting on embedded or retaining structures, and they can be also applied to the back analysis of geometry parameters, such as the dimensions of a soft inclusion or of a shallow cavity in a rock mass.

Various topics will be dealt with in the subsequent sections. First, the back analysis of the rock pressure acting on retaining structures is considered. Under the assumption of linear material behaviour these characterization problems are governed by a set of linear equations; therefore they are also referred to as 'linear' back analysis problems. Subsequently, the characterization of the material parameters of *in situ* rock mass is discussed. In this case the back analysis problem turns out to be nonlinear, even in the presence of linear material behaviour, and suitable minimization techniques have to be adopted for solution.

While the above topics are discussed in a deterministic context, a probabilistic viewpoint is considered for the third point, concerning the influence of the errors affecting the experimental measurements on the results of a back analyses. Finally, some applications of characterization techniques to actual engineering problems are illustrated in association with the monitoring of the stability of tunnels, and some comments are presented about possible further developments of the research in this field.

Notation. Boldface lower-case and capital letters denote, respectively, column vectors and matrices; superscripts T and -1 mean transpose and inverse.

BACK ANALYSIS OF THE ROCK PRESSURE

In most cases the back analysis of the rock pressure acting on embedded or retaining structures is carried out by adopting as input data the displacement components measured at selected points of the structure itself.

The solution of this problem can be reached on the basis of equations having the same order of those solving the corresponding stress analysis problems (i.e. of the equations leading to the displacements of the retaining structure once the rock pressure distribution is known). In particular, if the stress analysis is governed by a set of linear equations (as in the case of linearly elastic material behaviour and small displacement regime) also, the back analysis can be formulated in linear terms.

As a first, simple example consider the back analysis of the earth pressure acting on a vertical sheet pile. Assuming that the horizontal deflection y of the structure has been measured at various elevations z, an approximation of the deformed structural shape can be obtained, for instance, by the following sine and cosine series;

$$y(z) = \sum_{1}^{n} [a_i \sin(i\pi z/L) + b_i \cos(i\pi z/L)] \tag{1}$$

where L is the height of the structure. If the number m of measured displacements is greater than $2n$, the values of the unknown coefficients a_i and b_i can be obtained by a standard least square minimization. Having defined the deformed shape $y(z)$, it is possible to work out the earth pressure distribution $p(z)$ by means of the well-known equation of the elastic curve,

$$EJy^{IV}(z) = p(z) \tag{2}$$

which relates the unknown load distribution $p(z)$ to the fourth derivative of $y(z)$. In equation (2), EJ represents the flexural stiffness of the sheet pile.

This simple technique for back analysis presents the shortcoming of requiring several derivations of the function approximating the deformed shape. Since these derivations tend to increase the original error affecting the function $y(z)$, which in turn depends on the measurement error, the final load distribution may be barely reliable unless high precision measurements are used as input data.

A technique more sophisticated than the above one has been developed for the back analysis of the rock pressure on tunnel linings,[14,15] and is known as the integrated measuring technique. In order to apply this back analysis procedure it is necessary to subdivide the liner into segments of circular arch; the stretch L and the 'height' F of each segment are measured by means of high accuracy instruments in the undeformed (L and F) and deformed ($\bar{L}$ and $\bar{F}$) configurations (Figure 1), and the increments of these quantities, denoted by l and f, are calculated:

$$l = L - \bar{L}; \quad f = F - \bar{F} \tag{3a,b}$$

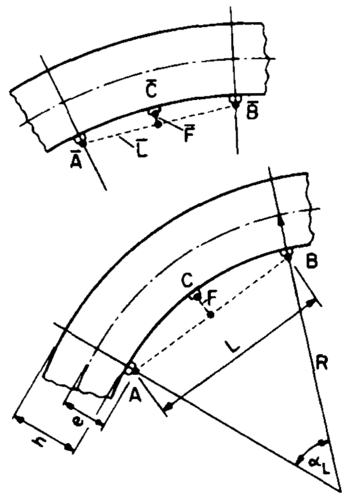


Figure 1. An arch segment in the undeformed (A, B and C) and deformed ($\bar{A}$, $\bar{B}$ and $\bar{C}$) state (after Reference 14)

If the angle α_L(Figure 1) is sufficiently small, which is an acceptable assumption in practice when the following inequalities hold,

$$R/h > 12 \quad \text{and} \quad R/L > 6 \tag{4a,b}$$

the increments of curvature χ and axial strain ε of each segment can be evaluated according to the following equations:

$$\chi = 8f/L^2; \quad \varepsilon = l/L + 8ef/L^2 \tag{5a,b}$$

In equations (4) and (5), R is the radius of the segment of circular arch, h is the thickness of the liner and e is the distance between the measurement point and the axis of the lining, as shown in Figure 1.

The distributions of l and f along the curvilinear co-ordinate s of the lining are arrived at by interpolating with suitable approximating functions their local values obtained from the field measurements. Then, the axial force N and the bending moment M in the linear are calculated by substituting these function in the following well-known relationships:

$$N(s) = \varepsilon(s)EA; \quad M(s) = \chi(s)EJ \tag{6a,b}$$

Finally, the normal p and shear t tractions exerted by the rock on the liner are obtained on the basis of the internal force distributions N and M, by imposing the equilibrium conditions of an infinitesimal portion of the arch (Figure 2):

$$p = \frac{N}{R} - \frac{\mathrm{d}^2 M}{\mathrm{d}s^2}; \quad t = \frac{1}{R}\frac{\mathrm{d}M}{\mathrm{d}s} + \frac{\mathrm{d}N}{\mathrm{d}s} \tag{7a,b}$$

An application of the integrated measuring technique concerning the Gotthard road tunnel has been presented in Reference 14. Some of the results of this study are reported in Figures 3 and 4 showing, respectively, the measurements performed along the liner and the results of back analysis.

The approaches discussed so far consider as input data only the displacement components of structural points. It has to be considered however that other data are sometimes available,

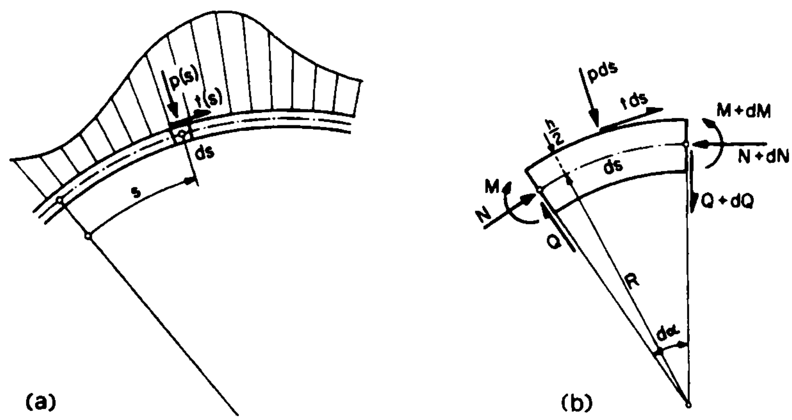


Figure 2. Acting internal and external forces: (a) a finite segment of the lining carrying a distributed rock load; (b) infinitesimal element for equilibrium considerations (after Reference 14)

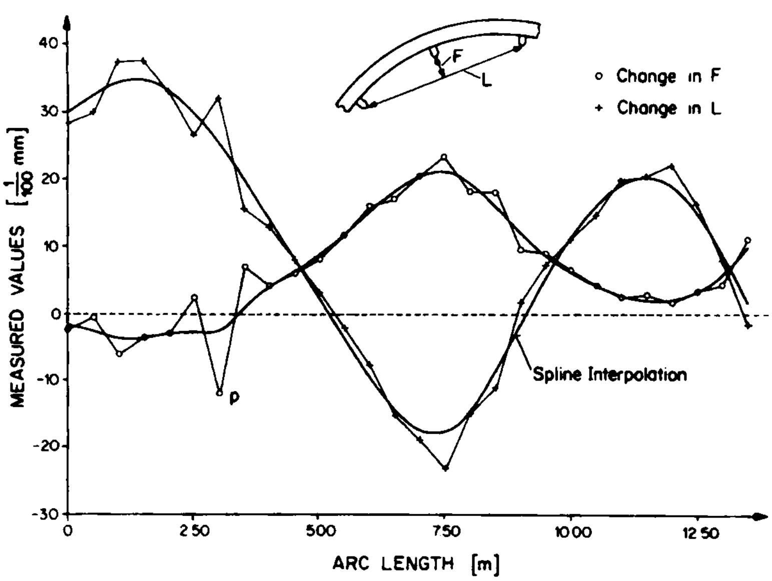


Figure 3. Distribution of the measured values F and L along the arch (after Reference 14)

concerning for instance the values of the earth pressure measured by pressure cells or values of concentrated loads due to anchors. A back analysis technique based on the finite element method and capable of taking into account displacements and other types of measured quantities has been presented in Reference 16. The use of this technique in the field of tunnel engineering was also suggested in Reference 17.

In order to illustrate the basis of the approach, consider the finite element mesh modelling a retaining structure and subdivide the elements into n_s sets or groups, so that the unknown earth pressure components normal and tangent to the elements of the ith set can be approximated by linear combinations of suitable functions:

$$\mathbf{q}_i = \mathbf{G}_i(\mathbf{x}_i)\mathbf{a}_i \tag{8}$$

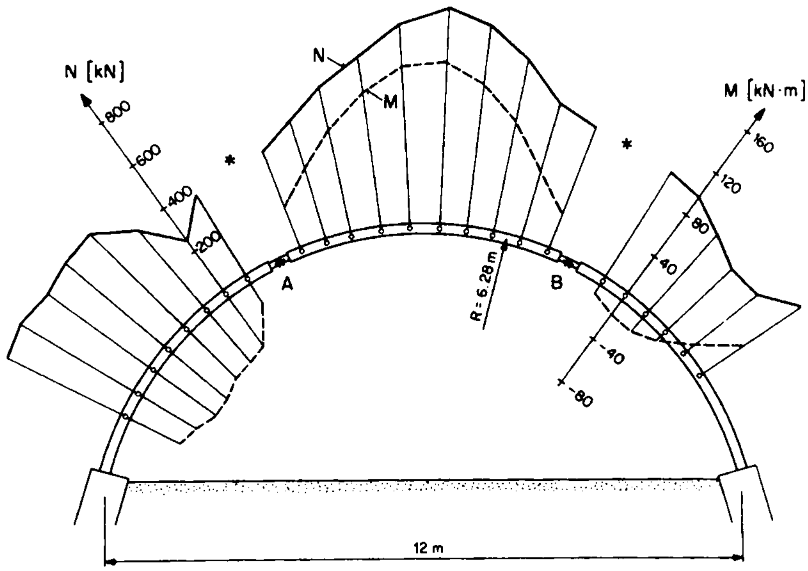


Figure 4. Distribution of the calculated moment and normal force along the arch (* = Load cell measurement of normal force N at A and B) (after Reference 14)

In equation (8) the index i denotes the element set, $\mathbf{q}$ is the vector of earth pressure components, $\mathbf{G}$ is the matrix of the approximating functions, $\mathbf{a}$ is the vector of unknown coefficients and $\mathbf{x}$ is the co-ordinate vector.

Denoting with $\mathbf{N}_i$ the matrix of the shape functions relating the displacements of the nodes of the ith element set to the displacement distributions within the elements, the nodal force vector equivalent to the earth pressure distribution is defined by the following integral:

$$\mathbf{f}_1 = \int_{A_i} \mathbf{N}_i^T \mathbf{T}_i \mathbf{q}_i \mathrm{d}A \tag{9}$$

Here, A_i is the area of the surface facing the ground of the ith element set and $\mathbf{T}_i$ is a transfer matrix relating the pressure components in the local reference frame (tangent and normal to the surface of the structure) to those in the global reference system.

Substituting equation (8) into equation (9), and writing equation (9) for all the sets of elements, a relation is obtained for the entire structure between the nodal forces $\mathbf{f}$ equivalent to the earth pressure and the unknown coefficients $\mathbf{a}$. This equation is expressed in the compact form,

$$\mathbf{f} = \mathbf{S}\mathbf{a} \tag{10}$$

where S is a matrix depending on the shape of the structure, on the characteristics of the adopted finite elements and of the pressure approximating functions.

If the performed *in situ* measurements provide a set of displacements and rotations of structural points (that coincides with nodes of the mesh), the well-known linear relationship between nodal forces $\mathbf{f}$ (or coefficients $\mathbf{a}$, through equation 10) and nodal displacements $\mathbf{u}$, through the stiffness matrix $\mathbf{K}$, can be partitioned as follows:

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & \mathbf{K}_{13} \\ \mathbf{K}_{21} & \mathbf{K}_{22} & \mathbf{K}_{23} \\ \mathbf{K}_{31} & \mathbf{K}_{32} & \mathbf{K}_{33} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_1^{\circ} \\ \mathbf{u}_2^{*} \\ \mathbf{u}_3 \end{Bmatrix} = \begin{bmatrix} \mathbf{S}_1 \\ \mathbf{S}_2 \\ \mathbf{S}_3 \end{bmatrix} \mathbf{a} \tag{11}$$

In this equation, vector $\mathbf{u}_1^o$ collects the displacement components that have to be constrained in order to eliminate all the rigid movements of the structure (hence, without losing generality, $\mathbf{u}_1^o = \mathbf{0}$), $\mathbf{u}_2^*$ is the vector of measured displacements and $\mathbf{u}_3$ collects the remaining (unknown) nodal displacements. A static condensation of equation (11) and some algebraic manipulations lead to the following equations:

$$\mathbf{Ca} = \mathbf{u}_2^*; \quad \mathbf{Da} = \mathbf{0} \tag{12.13}$$

where

$$\mathbf{C} = [\mathbf{K}_{22} - \mathbf{K}_{23}\mathbf{K}_{33}^{-1}\mathbf{K}_{32}]^{-1}[\mathbf{S}_2 - \mathbf{K}_{23}\mathbf{K}_{33}^{-1}\mathbf{S}_3] \tag{14}$$

$$\mathbf{D} = \mathbf{K}_{12}\mathbf{C} + \mathbf{K}_{13}\mathbf{K}_{33}^{-1}[\mathbf{S}_3 - \mathbf{K}_{32}\mathbf{C}] - \mathbf{S}_1 \tag{15}$$

Equations (12), (13) represent the first basic set of equations governing the back analysis problem; equation (12) relates the unknown parameters $\mathbf{a}$ to the measured displacements $\mathbf{u}_2^*$ and equation (13) ensures that the no 'fictitious' nodal reactions are generated at the nodes where the displacements $\mathbf{u}_1^o$ are defined.

In some cases it is possible to perform direct measurements of the earth pressure at some points of the structure (e.g. by means of pressure cells). These data, grouped in vector $\mathbf{q}^*$, can be expressed through equation (8) as linear combinations of some of the coefficients of vector $\mathbf{a}$:

$$\mathbf{La} = \mathbf{q}^* \tag{16}$$

Matrix $\mathbf{L}$ in equation (16) consists of parts of matrices $\mathbf{G}_i$ computed at the locations where the earth pressure components are measured.

Assembling together equations (12), (13) and (16), a system of linear equations is arrived at the solution of which, if the number of measured displacements and pressure is greater than the number of unknown parameters, can be based on the usual least square minimization:

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{L} \\ \mathbf{D} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{C} \\ \mathbf{L} \\ \mathbf{D} \end{bmatrix} \mathbf{a} = \begin{bmatrix} \mathbf{C} \\ \mathbf{L} \\ \mathbf{D} \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} \mathbf{u}_2^* \\ \mathbf{q}^* \\ \mathbf{0} \end{Bmatrix} \tag{17}$$

Note that additional conditions, expressed as linear combinations of the coefficients $\mathbf{a}$, could be considered in the problem formulation. For instance, one may impose continuity between the pressure distributions, and their derivatives, at the boundary between two adjacent groups of elements. Information concerning the location and direction of concentrated loads due to tie backs, measured stresses or strains at some points of the structure, etc., can also be taken into account. This back analysis technique has been applied to the determination of the earth pressure acting on a 28 m high diaphragm wall supported by three series of tie backs.

In situ measurements were performed at the end of construction in order to define the deformed configuration of the structure and the tensile forces on the anchors (Figure 5). On the basis of these data two back analyses were carried out. In both of them the structure was subdivided into four zones, the end points of which correspond to its connections with the anchors, and only the normal component of the earth pressure was considered as unknown. The first analysis assumes linear pressure distribution on each zone of the wall, while parabolic distributions were assumed in the second analysis. Additional conditions introduced in the calculations enforce the pressure continuity between two adjacent zones and zero pressure value at the upper end of the structure.

The results of analyses are shown in Figure 6, where the numerical results (dashed lines) are

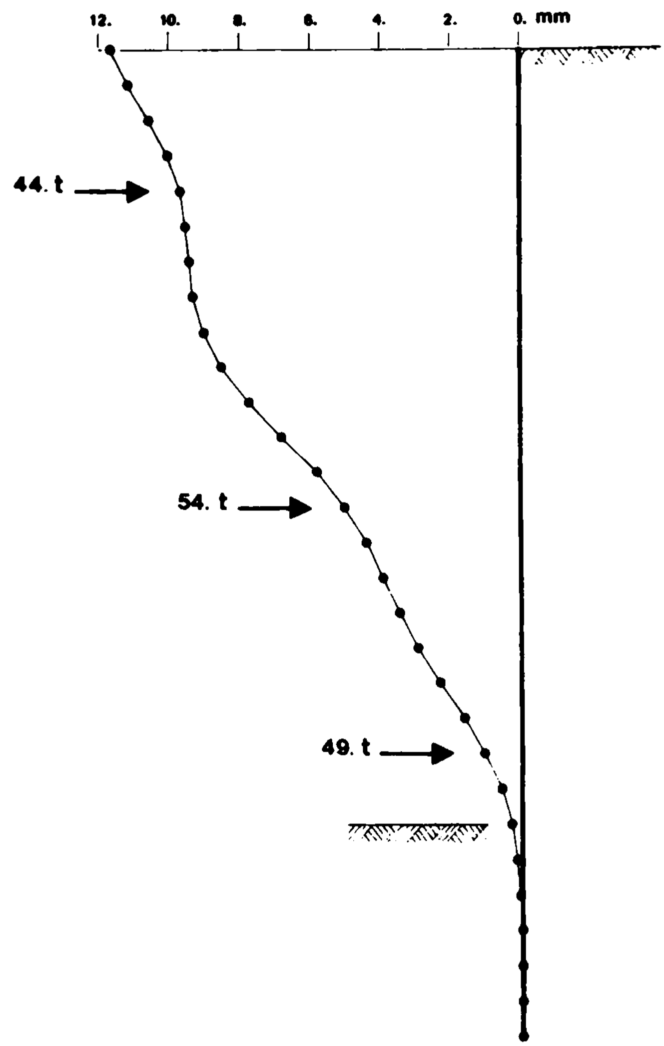


Figure 5. Measured displacements (dots) and anchor forces for a diaphragm wall (after Reference 16)

compared with those (solid lines) obtained on the basis of the classical solution proposed in Reference 18.

It is worth while observing that, regardless of the procedure adopted for back analysis, some requirements have to be fulfilled by the input data. A first 'quantitative' requirement concerns the number of measurements that has to be equal to or greater than the number of unknown parameters. In addition, various requirements on the 'quality' of the input data have to be fulfilled in order to obtain meaningful results from a back analysis. Among them a major role is played by the locations where the measurements are performed and by the direction in which the displacement are measured. For instance, it can easily be seen that the earth pressure distribution on the entire liner of a tunnel can hardly be determined on the basis of measurements performed in the vicinity of the tunnel crown only, regardless of their number. Note that equivalent

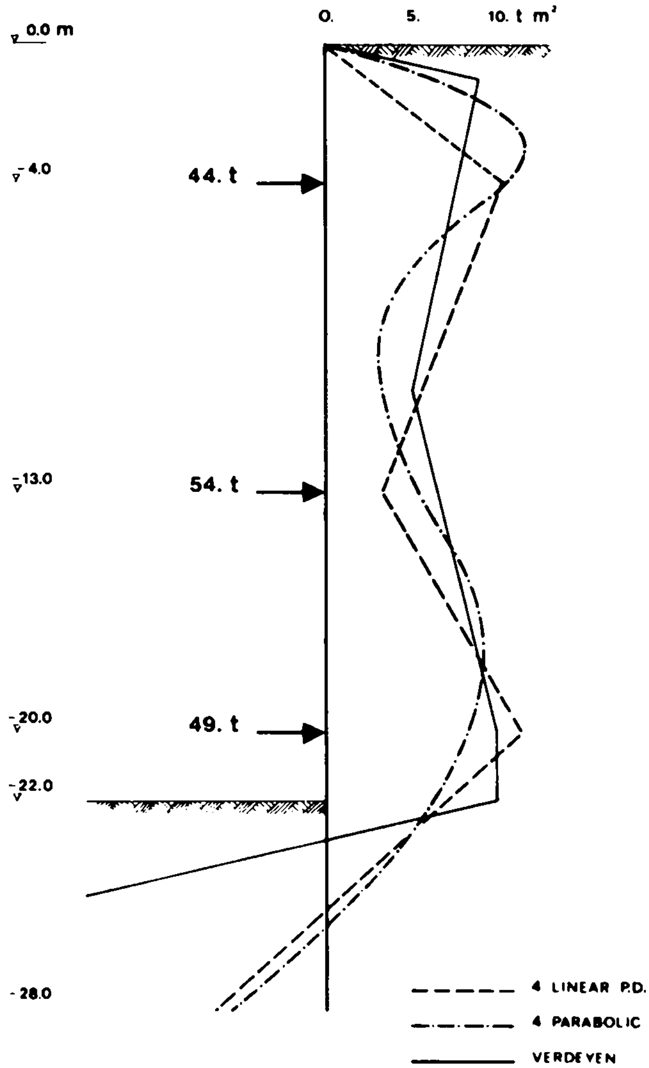


Figure 6. Comparison between back calculated (dashed lines) and analytical (solid line) earth pressure distribution (after Reference 16)

requirements about the quality[19] and the quantity of the input data exist also for the back analysis of material parameters which will be discussed in the next section.

BACK ANALYSIS OF MATERIAL PARAMETERS

As previously observed, the back analysis of mechanical parameters represents a nonlinear problem even in the simple case of linear elastic material behaviour. This can be shown by briefly recalling the basic characteristics of a technique for the back analysis of elastic constants[20,21] which is based on a finite element approch originally proposed in Reference 22 for structural engineering problems.

In order to apply this procedure it is necessary to establish a linear relationship between the stiffness matrix of each finite element $\mathbf{K}^e$ and the unknown material parameters. In the case of isotropic material behaviour, such a relationship can easily be obtained by describing the elastic behaviour in terms of bulk B and shear G moduli:

$$\mathbf{K}^e = B\mathbf{K}_B^e + G\mathbf{K}_G^e \tag{18}$$

The two matrices on the right-hand side of equation (18) represent, respectively, the volumetric and deviatoric stiffness of the eth element. Consequently, the stiffness matrix of the assembled finite element model can be written in the following form:

$$\mathbf{K} = \sum_{1}^{2n} p_i \mathbf{K}_i \tag{19}$$

where n is the number of different materials ($2n$ being the number of unknown elastic parameters) and $\mathbf{K}_i$ is the assembled stiffness matrix obtained by setting all the parameters to zero except for the ith parameter set equal to 1.

Assuming that m displacement components of points of the rock mass are measured in the field, and that these points coincide with nodes of the finite element mesh, the system of linear equations governing the behaviour of the finite element discretization can be partitioned as follows:

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_1^* \\ \mathbf{u}_2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{Bmatrix} \tag{20}$$

where vector $\mathbf{u}_1^*$ collects all the measured displacement components, and $\mathbf{f}_1$ and $\mathbf{f}_2$ are known nodal force vectors.

A static condensation of equation (20) leads to

$$(\mathbf{K}_{11} - \mathbf{Q}\mathbf{K}_{21})\mathbf{u}_1^* = \mathbf{f}_1 - \mathbf{Q}\mathbf{f}_2 \tag{21}$$

where

$$\mathbf{Q} = \mathbf{K}_{12}\mathbf{K}_{22}^{-1} \tag{22}$$

Taking into account equation (19), equation (21) can be written in the following form:

$$\sum_{1}^{2n} p_i \mathbf{r}_i = \mathbf{f}_i - \mathbf{Q}\mathbf{f}_2 \tag{23}$$

where

$$\mathbf{r}_i = (\mathbf{K}_{11,i} - \mathbf{Q}\mathbf{K}_{21,i})\mathbf{u}_1^* \tag{24}$$

The stiffness matrices in equation (24) are obtained by partitioning matrix $\mathbf{K}_i$ with the same criteria used in equation (20).

Grouping the unknown elastic parameters in the $2n$ vector $\mathbf{p}$, and grouping vectors $\mathbf{r}_i$ in the $m \times n$ matrix $\mathbf{R}$,

$$\mathbf{R} = [\mathbf{r}_1 | \mathbf{r}_2 | \cdots | \mathbf{r}_{2n}] \tag{25}$$

equation (23) yields the following relationship

$$\mathbf{R}\mathbf{p} = \mathbf{f}_1 - \mathbf{Q}\mathbf{f}_2 \tag{26}$$

that governs the calibration problem.

Assuming that the number of data (measured displacements) exceeds the number of unknowns (elastic constants), a standard least square minimization can be applied to equation (21), leading to the following nonlinear equation system (note in fact that the matrix of coefficients **R** depends, through matrix **Q**, on the unknown vector **p**):

$$\mathbf{R}^{\mathrm{T}}\mathbf{R}\mathbf{p} = \mathbf{R}^{\mathrm{T}}(\mathbf{f}_1 - \mathbf{Q}\mathbf{f}_2) \tag{27}$$

The solution of the above system is reached with a simple iterative procedure which requires at every step the inversion of a part ($\mathbf{K}_{22}$) of the assembled stiffness matrix, cf. equation (22), calculated on the basis of the parameters determined at the end of the preceding iteration.

Other approaches for the back analysis of elastic parameters have been proposed (see, for example, References 23 and 24) which have less general applicability, but require also less programming effort than the one previously described. For instance, the one proposed in Reference 24 is applicable only to the determination of the elastic constants (Young's modulus E and Poisson's ratio ν) of homogeneous rock masses, but offers the advantage of not requiring the implementation of *ad hoc* computer codes.

The basis of this technique can be summarized as follows. A series of finite element analyses is performed, assuming $E = 1$ and varying the values of Poisson's ratio, obtaining one set of nodal displacements for each of them. Let us denote with ρ_i^{field} the displacement components measured in the field ($i = 1, m$) and with $\rho_i^{FE}(\nu)$ the corresponding values obtained by the finite element solution based on a given Poisson's ratio.

On these bases, it is possible to determine one 'optimal' value of Young's modulus for each analysis (i.e. for each value of ν) and for each measurement point by means of the following simple equation:

$$E_i(\nu) = \rho_i^{\mathrm{FE}}(\nu)/\rho_i^{\mathrm{field}} \tag{28}$$

The results can be plotted as shown in Figure 7(b), where the curves refer to the vertical v and horizontal u displacements caused by a trapezoidal load distributions at various points of homogeneous rock mass (see Figure 7a). If all curves meet approximately at the same point, the optimal values of the elastic constants for the entire rock mass are readily determined.

In addition to its notable simplicity, this technique presents the advantage of providing an indication about suitable locations for the measurements points. This is obtained by examining the shape of the E–ν curves. If two of these curves have quite similar shapes, the two corresponding measurements provide almost the same information and consequently one of them could be omitted from the measurement program.

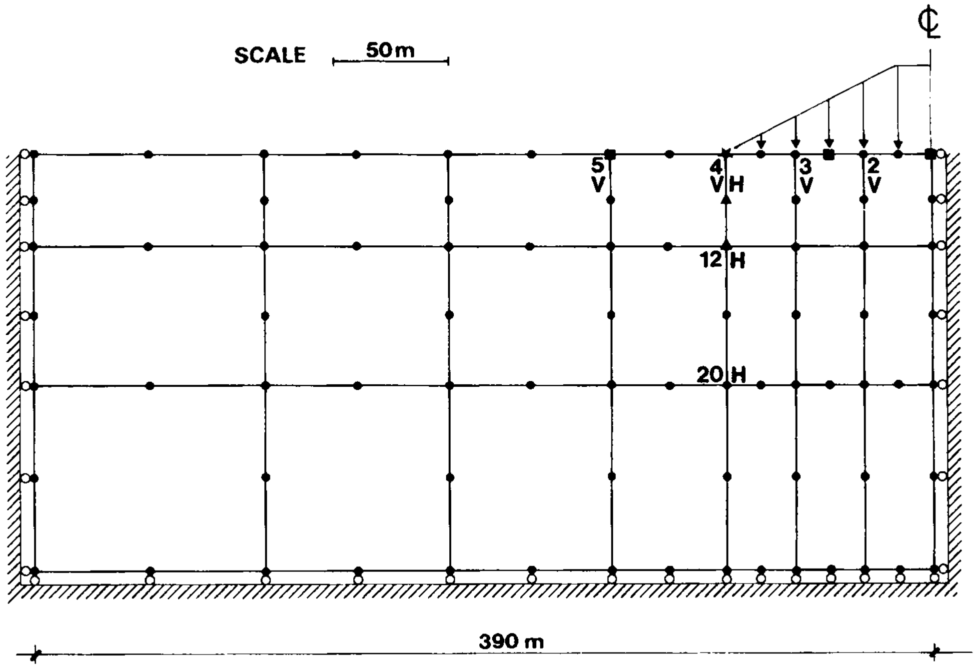
It has to be considered, however, that this method would not lead to useful results when the E–ν curves do not intersect each other almost at the same point. This is likely to happen, for instance, when non-negligible experimental errors affects the measured displacements.

An alternative procedure, with respect to the previous one, can be based on the direct minimization of the discrepancy between the field measurements and the corresponding numerically evaluated quantities. This approach presents the advantage of avoiding the 'inversion' of the stress analysis equations which was required by the above technique operating through the least squares method.

The following error function can be adopted as a practical definition of the discrepancy between the measured displacements (denoted by an asterisk) and those deriving from a numerical stress analysis in which a given set of material parameters **p** is adopted:

$$\mathrm{Err} = \sum_{1}^{m} [u_i^* - u_i(\mathbf{p})]^2 \tag{29}$$

Clearly, other definitions are possible, for instance considering only the maximum absolute value



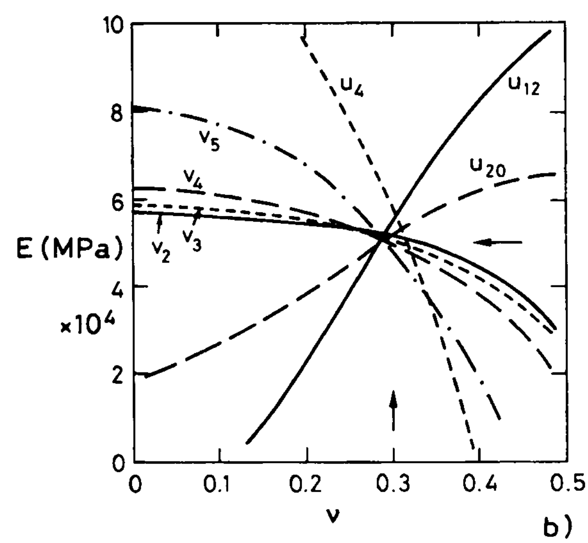


Figure 7. (a) Finite element mesh and load distribution for the back analysis; (b) variation of the back calculated elastic modulus with the estimated Poisson's ratio (after Reference 24)

of the differences in equation (29), or dividing the terms of the summation by the corresponding measured value in order to obtain a non-dimensional definition of the error.

Since the error function depends through the numerical results on the parameters to be back calculated (which in this context have a rather general meaning and may correspond to elasticity or shear strength properties, viscosity coefficients, etc.), the back analysis reduces to determining the set of parameters that minimizes the error function, i.e. that leads to the best approximation of the field observation through the chosen numerical model.

The error defined by equation (29) is in general a complicated nonlinear function of the unknown quantities, and in most cases the analytical expression of its gradient cannot be determined. This is particularly evident for nonlinear or elasto-plastic problems. Therefore, the adopted minimization algorithm must handle general nonlinear functions and it should not require the analytical evaluation of the function gradient.

A comprehensive discussion of the algorithms fulfilling the above requirements, known in mathematical programming as the direct search methods, can be found for example, in Reference 25. These are iterative procedures which perform the minimization process only by successive evaluations of the error function. In the present contest, each evaluation requires a stress analysis of the geotechnical problem on the basis of the trial vector $\mathbf{p}$ chosen for that iteration. Some comments on the use of the direct search method for back analyses in the field of geomechanics have been presented in Reference 26.

In most practical cases some limiting values exist for the unknown parameters. For instance, the modulus of elasticity cannot reach negative values. These limits, expressed by inequality constraints, can easily be introduced into a direct search algorithm by means of a penalization procedure. When a point in the space of the free variables is reached outside the feasible domain, the error function is assigned a large value so that the minimization algorithm automatically drives back the optimization path into the feasible region. This penalty approach turns out to be general and simple to implement. In fact, no assumptions are required on the characteristics of the constraints (e.g. about their convexity) and the computer program for constrained minimization can easily be obtained with few modifications of the corresponding unconstrained code.

From the computational viewpoint, the back analysis approach requiring the minimization of the error function expressed by equation (29) presents non-negligible differences with respect to that based on the least square method. It turns out, in fact, that the least square technique, specifically developed for the calibration of elasticity constants, converges towards the optimal values of the parameters faster and 'smoother' than the direct search procedure. As a consequence, the computer cost required by the first solution method is in general smaller than that of the second one.

The different performance of the two approaches can easily be seen, for instance, by applying both of them to the back analysis of the elastic constants (bulk modulus B and shear modulus G) of the rock underlying the previously mentioned embankment. The results of such analyses are summarized by the diagrams in Figure 8, showing the values of B and G obtained at each iteration of the solution process. The direct search algorithm adopted for this example is the so-called 'Simplex' method presented in Reference 27.

It has to be recognized, however, that back analysis procedure based on direct search algorithm present also a non-negligible advantage with respect to the least squares approach. In fact, while the least squares procedure requires the implementation of an *ad hoc* computer program, the direct search approaches can be developed on the basis of standard computer codes for nonlinear function minimization in which the finite element program for stress analysis is introduced as a subroutine. This requires some simple changes of the original finite element code and a limited

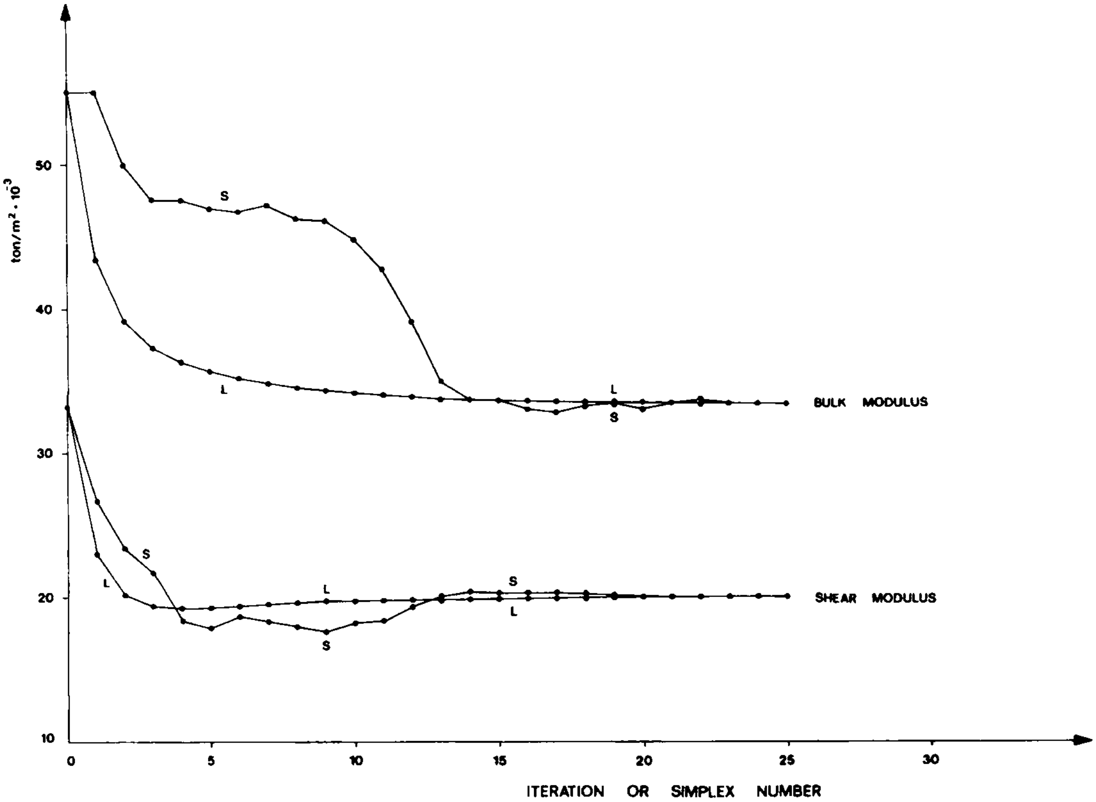


Figure 8. Variation of the bulk and shear moduli during the error minimization process. (*L* refers to the least squares algorithm; *S* refers to the Simplex algorithm) (after Reference 21)

programming effort. In addition, the same stress analysis and error minimization programs can be used for various characterization problems, merely by considering the calculated quantities as functions of the unknown parameters, regardless of their nature.

For instance, direct search algorithms have been applied to the calibration of elasto-plastic and visco-plastic material models for *in situ* rock masses[28,29] and to back analyses related to slope stability problems.[30]

Another technique for the back analysis of elasticity parameters has been proposed in References 31 and 32, where the conjugate gradient method[33] is adopted for minimizing the error function in equation (29). Also in this case the final solution is reached by means of an iterative process, but while the previously mentioned direct search algorithms require at every iteration only the evaluation of the error function, the gradient type techniques require also the determination of the function derivatives with respect to the free variables. Under the assumption of linear material behaviour, the analytical expression of these derivatives can be worked out and directly programmed into the computer code developed for the solution of the calibration problem.

The results obtained in the solution of some significant examples show that this method is characterized by a fast convergence. This should compensate the additional computer effort, with respect to the direct search approaches, required by the gradient evaluation, leading to an overall performance comparable to that of the least squares technique. A comparative evaluation of these solution approaches is presented in Reference 34.

The same conjugate gradient method was also adopted for the back analysis of the elasticity and permeability constants of soft clay deposits.[35] The calibration is based on the 'coupled' finite element approach for consolidation analysis in which the simultaneous problems of deformation of the soil skeleton and seepage of the pore fluid are solved. In this case, a term was added to the error function in equation (29) in order to account for the discrepancy between calculated and measured pore pressures.

It can be observed that a significant amount of research has been devoted to the back analysis of the mechanical properties governing the consolidation of natural clay deposits (see, for example, References 36–40). This is probably due to two main causes. First, it is well known that laboratory tests on small-scale samples can hardly provide meaningful values of some parameters (such as the coefficients of permeability) characterized by a marked space variation in the field. Consequently, it seems preferable to determine their 'average' *in situ* value through the back analysis of field measurements.

In addition, these measurements performed at various times during the consolidation process can be used to refine the numerical model adopted in the calculations. This permits, in turn, to improve with time the quality of important engineering predictions concerning, for instance, the value of the final settlements or of the time necessary to reach a given percentage of it.

A PROBABILISTIC APPROACH TO BACK ANALYSIS

The use of probabilistic approaches for the calibration of numerical models is relatively frequent in the field of geomechanics (see, for example, References 41–44. In fact, important practical aspects can be taken into account by means of these approaches, such as: the uncertainties related to the definition of the soil layers or of the space variation of the mechanical properties; the influence of the back calculated parameters of the experimental error affecting the field measurements; etc.

Among various alternative probabilistic procedures, the so-called Bayesian approach[45,46] presents non-negligible advantages.[47–49] First, the subjective judgement of an expert, or other valuable *a priori* information on the parameters to be back calculated, can be taken into account in addition to the experimental data. Note that this is quite an important point when dealing with geomechanics problems. In many cases, in fact, the opinion of an experienced consultant has a 'weight' comparable to that of costly *in situ* measurement program. It can be also observed that the above feature makes the Bayesian calibration methods similar, in a sense, to the reasoning process followed by the designer who, combining his experience with the data from the field, formulates a reasonable guess on the values of the parameters to be adopted in the calculations.

A second important characteristic of these methods is that the estimation of the unknown parameters can be 'updated' if additional experimental information becomes available during time. This turns out to be extremely useful when dealing with time-dependent (e.g. consolidation) processes, i.e. when subsequent sets of field measurements can be performed in order to obtain a continuous refinement of the estimation of the soil properties.

Recently,[50] the Kalman filter theory[51] has been proposed as an alternative probabilistic approach to calibration analyses in soil mechanics.

In the following the discussion is limited to a brief outline of a Bayesian approach for the back analysis of material parameters in the presence of data affected by experimental errors. It can be mentioned that other methods are applicable to this type of analysis, like the 'simulation' technique adopted in Reference 52, based on formulations simpler than the one here discussed. However, these methods usually require a computational effort much larger than that necessary with a Bayesian procedure, especially when dealing with a relatively large number of free variables, and this strongly limit their use in the solution of practical problems.

Consider the experimental measurements $\mathbf{u}^*$ as affected by errors, which are grouped in vector $\Delta\mathbf{u}$ and are seen as random variables, and assume that the expected average value of the error (expressed by means of the 'expectation' operator $E|\cdot|$) vanishes:

$$E|\Delta\mathbf{u}| = \mathbf{0} \tag{30}$$

Assume also that the error covariance matrix $\mathbf{C}_u$, that depends on the measuring device accuracy, is known:

$$\mathbf{C}_u = E|\Delta\mathbf{u}\,\Delta\mathbf{u}^{\mathrm{T}}| \tag{31}$$

If all measurements are statistically independent, $\mathbf{C}_u$ is a diagonal matrix the entries of which (variances) are related to the resolution of the instruments.

Also the unknown parameters $\mathbf{p}$ are regarded as random quantities and it is assumed that the following expectations are known:

$$\mathbf{p}_o = E|\mathbf{p}|; \quad \mathbf{C}_p^o = E|[\mathbf{p} - \mathbf{p}_0][\mathbf{p} - \mathbf{p}_o]^{\mathrm{T}}| \tag{32a,b}$$

In the above equation, $\mathbf{p}_o$ and $\mathbf{C}_p^o$ represent the *a priori* information on the unknown parameters, deriving from the experience of the engineer or from previous back analyses of different set of experimental data. If the entries of vector $\mathbf{p}_0$ are uncorrelated, $\mathbf{C}_p^o$ is a diagonal matrix. Note that the values of the entries of this matrix increase with decreasing 'reliability' of the initial information on the unknown parameters.

The Bayesian back analysis consists in combining the *a priori* and the experimental information, in order to achieve the best estimate of the unknown parameters. Also in this case, as in a deterministic back analysis, a numerical model is necessary which is used to calculate the quantities $\mathbf{u}$, corresponding to the measured ones $\mathbf{u}^*$, on the basis of a trial parameter vector $\mathbf{p}$.

Consider first the simple case in which $\mathbf{u}$ is linearly dependent on $\mathbf{p}$, through a constant matrix $\mathbf{L}$ and constant vectors $\mathbf{u}'$ and $\mathbf{p}'$:

$$\mathbf{u}(\mathbf{p}) = \mathbf{u}' + \mathbf{L}\{\mathbf{p} - \mathbf{p}'\} \tag{33}$$

The best estimate $\bar{\mathbf{p}}$ of $\mathbf{p}$ can be obtained by minimizing with respect to the parameter vector the following error function:

$$\mathrm{Err} = [\mathbf{u}^* - \mathbf{u}(\mathbf{p})]^{\mathrm{T}}[\mathbf{C}_u]^{-1}[\mathbf{u}^* - \mathbf{u}(\mathbf{p})] + [\mathbf{p}_o - \mathbf{p}]^{\mathrm{T}}[\mathbf{C}_p^o]^{-1}[\mathbf{p}_o - \mathbf{p}] \tag{34}$$

which consists of two parts: the first represents the discrepancy between measured and calculated data and the second is the discrepancy between assumed and current parameters. The discrepancies are weighted by means of the inverted covariance matrices which tend to vanish with decreasing accuracy, or reliability, of the experimental data and of the *a priori* infomation.

By introducing equation (33) into equation (34), and by imposing that the derivatives of the error function with respect to $\mathbf{p}$ vanish, the following system of linear equations is arrived at:

$$[\mathbf{L}^{\mathrm{T}}\mathbf{C}_u^{-1}\mathbf{L} + (\mathbf{C}_p^o)^{-1}]\mathbf{p} = \mathbf{L}^{\mathrm{T}}\mathbf{C}_u^{-1}[\mathbf{u}^* - \mathbf{u}' + \mathbf{L}\mathbf{p}'] + [\mathbf{C}_p^o]^{-1}\mathbf{p}_o \tag{35}$$

the solution of which leads to the optimal vector $\bar{\mathbf{p}}$:

$$\bar{\mathbf{p}} = [\mathbf{I} - \mathbf{M}_o\mathbf{L}]\mathbf{p}_o + \mathbf{M}_o\mathbf{u}^* - \mathbf{M}_o[\mathbf{u}' - \mathbf{L}\mathbf{p}'] \tag{36}$$

In the above equation, $\mathbf{I}$ is the identity matrix and

$$\mathbf{M}_o = [\mathbf{L}^{\mathrm{T}}\mathbf{C}_u^{-1}\mathbf{L} + (\mathbf{C}_p^o)^{-1}]^{-1}\mathbf{L}^{\mathrm{T}}\mathbf{C}_u^{-1} \tag{37}$$

In order to obtain the covariance matrix associated with vector $\bar{\mathbf{p}}$, it is necessary to recall that if a vector, say $\mathbf{a}$, is linearly dependent on a vector $\mathbf{b}$ of random variables through a constant matrix $\mathbf{A}$,

$$\mathbf{a} = \mathbf{A}\mathbf{b} \tag{38}$$

the following relationship exists between the covariance matrices, $\mathbf{C}_a$ and $\mathbf{C}_b$, associated with the two vectors:

$$\mathbf{C}_a = \mathbf{A}\mathbf{C}_b\mathbf{A}^{\mathrm{T}} \tag{39}$$

Due to the facts that a linear relationship exists between vectors $\bar{\mathbf{p}}, \mathbf{p}_o$ and $\mathbf{u}^*$ (cf. equation 36), and that $\mathbf{p}_o$ and $\mathbf{u}^*$ are statistically uncorrelated, the following expression can be established on the basis of equation (39) for the covariance matrix $\mathbf{C}_p$ associated with the optimal vector $\bar{\mathbf{p}}$:

$$\mathbf{C}_p = [\mathbf{I} - \mathbf{M}_o\mathbf{L}]\mathbf{C}_p^o[\mathbf{I} - \mathbf{M}_o\mathbf{L}]^{\mathrm{T}} + \mathbf{M}_o\mathbf{C}_u\mathbf{M}_o^{\mathrm{T}} \tag{40}$$

It is important to point out that equations (36) and (40) are not directly applicable to the majority of calibration problems in the field of geomechanics, due to the fact that $\mathbf{u}$ is in general a nonlinear function of $\mathbf{p}$. In order to solve the nonlinear back analysis problem an iterative procedure can be adopted, based on the linearization of the $\mathbf{u}$–$\mathbf{p}$ relationship in the neighbourhood of the current parameter vector $\mathbf{p}'$ by means of a Taylor's series expansion truncated at the linear terms (cf. equation 33):

$$\mathbf{u}(\mathbf{p}) \simeq \mathbf{u}(\mathbf{p}') + \mathbf{L}(\mathbf{p}')\{\mathbf{p} - \mathbf{p}'\} \tag{41}$$

The main steps of the iterative solution procedure for the nonlinear problem can be summarized as follows:

1. The current parameter vector $\mathbf{p}'$ is set equal to its initial estimate $\mathbf{p}_o$,
2. The quantities $\mathbf{u}' = \mathbf{u}(\mathbf{p}')$ are determined by means of a stress analysis of the chosen numerical model.
3. The current 'sensitivity' matrix $\mathbf{L}(\mathbf{p}')$ is evaluated numerically as a finite difference approximation. This implies to solve n stress analysis problem (n being the number of unknown parameters p_i). The vector of parameters used in each analysis coincides with vector $\mathbf{p}'$, except for the ith component which is perturbed by a small quantity Δp_i. Denoting with Δu_i the difference between the quantities obtained at step 2 and those derived from the ith 'perturbed' analysis, the sensitivity matrix can be expressed as follows:

$$\mathbf{L}(\mathbf{p}') = [\Delta\mathbf{u}_1/\Delta p_1 | \Delta\mathbf{u}_2/\Delta p_2 | \cdots | \Delta\mathbf{u}_n/\Delta p_n] \tag{42}$$

4. Vector $\bar{\mathbf{p}}$ is evaluated by means of equation (36), where the current values of $\mathbf{L}, \mathbf{u}'$ and $\mathbf{p}'$ are introduced.
5. The iterations end when the difference between $\mathbf{p}'$ and $\bar{\mathbf{p}}$ becomes smaller than a pre-assigned tolerance, otherwise $\mathbf{p}'$ is set equal to $\bar{\mathbf{p}}$ and the process is continued from step 2.

When convergence is reached, the final covariance matrix $\mathbf{C}_p$ is calculated by means of equation (40). The main diagonal of this matrix contains the variances which define the uncertainties of the estimated values of the parameters. It is worth observing that, when a reliable initial guess on the parameters can be formulated, the Bayesian calibration approach is applicable also if the number of unknown parameters exceeds the number of *in situ* measurements.

Consider in fact the limit case in which no experimental information is available. This case can be seen as equivalent to the situation in which the accuracy of the experimental data is so poor that the entries of the corresponding inverted covariance matrix $\mathbf{C}_u$ vanish. Consequently, equation (36) reduces to the trivial form expressing the expected equivalence between the optimal values of the parameters $\bar{\mathbf{p}}$ and their initial estimate $\mathbf{p}_o$.

Another limit case is when no *a priori* information is available, or when its reliability is so low that the corresponding inverted covariance matrix coincides with the zero matrix. In this case equation (35) becomes

$$[\mathbf{L}^T\mathbf{C}_u^{-1}\mathbf{L}]\mathbf{p} = \mathbf{L}^T\mathbf{C}_u^{-1}[\mathbf{u}^* - \mathbf{u}' + \mathbf{L}\mathbf{p}'] \tag{43}$$

Furthermore, if all the (uncorrelated) *in situ* measurements have the same accuracy, matrix $\mathbf{C}_u$ can be eliminated from equation (43), thus obtaining the following least square expression for the best estimate of the unknown parameters:

$$\mathbf{L}^T\mathbf{L}\mathbf{p} = \mathbf{L}^T[\mathbf{u}^* - \mathbf{u}' + \mathbf{L}\mathbf{p}'] \tag{44}$$

Note, however, that the covariance matrix of the measurements still affects the covariance matrix associated with the best estimate of the parameters. In fact, equation (40) becomes

$$\mathbf{C}_p = \mathbf{M}_o\mathbf{C}_u\mathbf{M}_o^T \tag{45}$$

where

$$\mathbf{M}_o = [\mathbf{L}^T\mathbf{C}_u^{-1}\mathbf{L}]^{-1}\mathbf{L}^T\mathbf{C}_u^{-1} \tag{46}$$

An application of this probabilistic approach has been discussed in Reference 53, with reference to the back analysis of the mechanical characteristics of a soft clay deposit, underlying a railroad embankment, in which sand drains were installed in order to increase the rate of the consolidation process. The available data consists of settlements and pore pressures measured during time in the vicinity of the embankment. The back analysis was performed by subdividing the soil mass into zones (Figure 9) having different mechanical properties, and considering as unknowns the permeability coefficients (in the vertical and horizontal directions) and the elastic modulus for each zone.

Two problems were solved separately adopting the same finite element discretization. The first one, based on the 'long term' (or final) values of the *in situ* measurements, led to the elastic moduli for the various zones of the deposit. The second problem concerned the determination of the permeability coefficients and was based on the entire set of data collected during time.

The entries of the diagonal covariance matrix $\mathbf{C}_u$ associated with the *in situ* measurements was defined considering both the accuracy of the measuring devices and the lack of information on the seasonal variation of the water table level. The initial values of the permeability coefficients were determined by simple calculations, based on the results of laboratory tests on clay samples, in which the sand drains were introduced in an approximated way. The corresponding initial variances accounted for the large uncertainty of this initial estimation.

In Figure 10 a comparison is shown between some of the measurements performed *in situ* during time and the numerical simulation of the construction process of the embankment, based on the back calculated parameters of the clay deposit. A satisfactory agreement can be observed between experimental and numerical results, but for the pore pressures. In fact, the data from the installed piezometers were affected by seasonal variations of the water table that were not known with sufficient accuracy.

As previously observed, in addition to the optimal values of the parameters, the Bayesian method of analysis provides also the associated final covariance matrix that permits a quantitative evaluation of the reliability of the results of back analysis. For the problem at hand, this additional information shows that (a) the horizontal coefficients of permeability are determined with an accuracy higher than that obtained for the vertical ones, and (b) the accuracy of the elasticity moduli of the soft zones (having a marked influence on the settlements) is larger than that of the relatively hard ones.

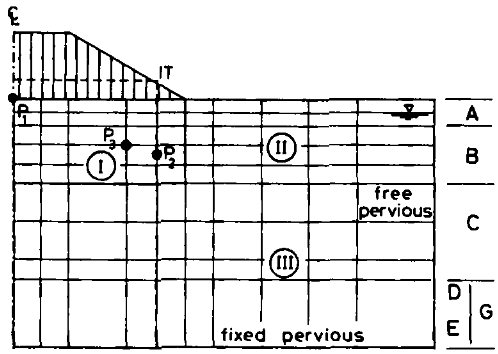


Figure 9. Finite element mesh and boundary conditions (zone I represents the zone with vertical sand drains) (after Reference 53)

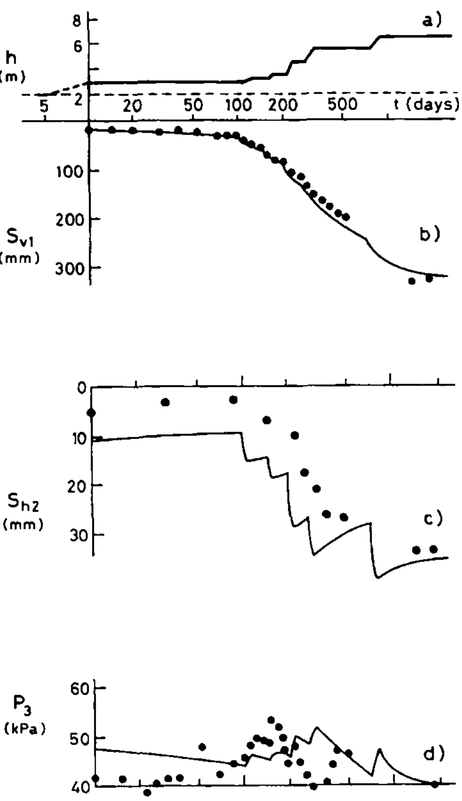


Figure 10. Consolidation analysis: (a) embankment height; (b) surface settlement (at point P_1 in Figure 9); (c) lateral movement (at point P_2 in Figure 9); (d) pore water pressure (at point P_3 in Figure 9) (black dots = experimental data) (after Reference 53)

AN APPLICATION OF BACK ANALYSIS IN TUNNELLING ENGINEERING

One of the most important and commonly met problems in tunnelling engineering concerns the assessment of stability of underground openings on the basis of displacements measured during the excavation works. The various methods adopted in practice for solving this problem require as input data the *in situ* stress state and the mechanical properties of the rock mass. Unfortunately, the simultaneous evaluation of both parameter sets on the basis of the measured displacements is not an easy task. In fact, the standard procedures for the interpretation of field measurements require either known initial stresses, in order to determine the material constants, or known material parameters, in order to evaluate the initial stress state.

It has to be considered, however, that the determination of the 'true' initial stresses and rock mass properties would not be necessary, if the back analysis of the available field measurements could provide an alternative, reliable way for evaluating the stability of excavation.

Based on the above considerations, a back analysis method has been presented in References 54–56 for monitoring the stability of underground openings, which rests on the concepts of 'normalized' *in situ* stresses and 'critical strain' for the rock mass and does not need the separate determination of the two above-mentioned sets of parameters. A typical feature of this approach is the relatively small amount of calculations required. This is an important point since the back analysis, in order to be effective, should be simple enough to permit the interpretation of the measured data in a limited time and directly at the construction site, e.g. through the use of a microcomputer.

The main assumptions on which this procedure rests are summarized in the following points:

1. The deformational behaviour of the rock mass is idealizd by a homogeneous, isotropic and linearly elastc material, so that the mechanical constants reduce to Young's modulus E and Poisson's ratio ν. Since ν has the least influence on the results of stress analyses, an appropriate value can be chosen for it and adopted in the calculations.
2. The elastic constants of the lining are known.
3. The *in situ* stress state linearly increases with depth due to the own weight of the rock, while it remains constant along the axis of the tunnel. Based on these assumptions, the problem can be treated in two-dimensional, plane strain, conditions and only three components of the *in situ* stress state $\boldsymbol{\sigma}^o$ can be considered:

$$\boldsymbol{\sigma}^o = \{\sigma_x^o \; \sigma_y^o \; \tau_{xy}^o\}^{\mathrm{T}} \tag{47}$$

The variation due to the excavation of the stress state in the rock mass can be evaluated through a finite element analysis in which equivalent nodal forces $\mathbf{f}^o$, corresponding to the release of the initial stresses, are applied to the contour of the opening. These forces can be expressed in integral form,

$$\mathbf{f}^o = \sum_i \left(\int_{V_i} \mathbf{B}^{\mathrm{T}} \boldsymbol{\sigma}^o \mathrm{d}V + \int_{V_i} \mathbf{N}^{\mathrm{T}} \boldsymbol{\gamma} \, \mathrm{d}V \right) \tag{48}$$

where the summation goes over the elements facing the excavation contour, $\mathbf{N}$ and $\mathbf{B}$ are the matrices of the element shape functions and of their derivatives, γ is the vector of the body force components due to gravity, and V_i is the volume of the ith finite element.

The stiffness matrix $\mathbf{K}$ of the finite element mesh that discretizes both the liner and the surrounding rock can be expressed as

$$\mathbf{K} = E_R \mathbf{K}^* \tag{49}$$

where

$$\mathbf{K}^* = \mathbf{K}_R + R\mathbf{K}_L; \quad R = E_L/E_R \tag{50a, b}$$

In the above equations, E_R and E_L are the elastic moduli of rock and lining; $\mathbf{K}_R$ and $\mathbf{K}_L$ are the stiffness matrices of the portions of the mesh corresponding to the rock mass and the liner evaluated, respectively, assuming unit values for E_R and E_L. Note that the coefficient R can be considered equal to zero if the liner is not installed, or if its stiffness is negligible with respect to that of the rock.

Taking into account equations (47), (48) and (49), the liner relationship between nodal displacements $\mathbf{u}$ and nodal forces, through the stiffness matrix $\mathbf{K}$, can be written in the following convenient form:

$$\mathbf{K}^*\mathbf{u} = \frac{\sigma_x^o}{E_R}\mathbf{f}_x^o + \frac{\sigma_y^o}{E_R}\mathbf{f}_y^o + \frac{\tau_{xy}^o}{E_R}\mathbf{f}_{xy}^o \tag{51}$$

Here , $\mathbf{f}_x^o$, $\mathbf{f}_y^o$ and $\mathbf{f}_{xy}^o$ are 'equivalent' force vectors corresponding to unit values of each initial stress components in turn.

Let us define $\mathbf{u}_x$ as the displacement vector obtained by equation (51) considering $\sigma_x^o = 1$ and all other stress components equal to zero. Obvious equivalent definitions hold for vectors $\mathbf{u}_y$ and $\mathbf{u}_{xy}$. Let us also define the 'normalized' initial stresses $\boldsymbol{\sigma}^*$ as the ratio between the *in situ* stresses and the elastic modulus of the rock mass:

$$\boldsymbol{\sigma}^* = \left\{ \frac{\sigma_x^o}{E_R} \quad \frac{\sigma_y^o}{E_R} \quad \frac{\tau_{xy}^o}{E_R} \right\}^{\mathrm{T}} \tag{52}$$

Then, a linear relationship can be established between the displacements $\mathbf{u}$ and the stresses $\boldsymbol{\sigma}^*$;

$$\mathbf{U}\boldsymbol{\sigma}^* = \mathbf{u} \tag{53}$$

where

$$\mathbf{U} = [\mathbf{u}_x \quad \mathbf{u}_y \quad \mathbf{u}_{xy}] \tag{54}$$

The displacement components that do not correspond to measured quantities in the field can be eliminated from equation (53). Let us assume that this reduction has been already done and consider, in what follows, that equation (53) concerns only the displacement components determined *in situ*. Since in practice only relative displacements $\Delta\mathbf{u}$ can be easily evaluated, which are related to the absolute ones $\mathbf{u}$ by a suitable transformation matrix $\mathbf{T}$,

$$\Delta\mathbf{u} = \mathbf{T}\mathbf{u} \tag{55}$$

it is convenient to rewrite equation (53) in terms of the measured relative displacements:

$$\mathbf{T}\mathbf{U}\boldsymbol{\sigma}^* = \Delta\mathbf{u} \tag{56}$$

If the number of measured displacement $\Delta\mathbf{u}$ is greater than the number of the components of vector $\boldsymbol{\sigma}^*$ (which is equal to 3, cf. equation 52), a least square minimization can be applied to equation (56), obtaining the following system of three linear equations:

$$[\mathbf{TU}]^{\mathrm{T}}[\mathbf{TU}]\boldsymbol{\sigma}^* = [\mathbf{TU}]\Delta\mathbf{u} \tag{57}$$

The solution of equation (57) leads to the normalized initial stresses $\boldsymbol{\sigma}^*$, that represents the final results of the back analysis. On the basis of these results it is possible to assess quantitatively the stability of the opening and the overall conditions of the rock surrounding it.

Consider in fact the case in which a reasonable value of the *in situ* vertical stress σ_y^o can be

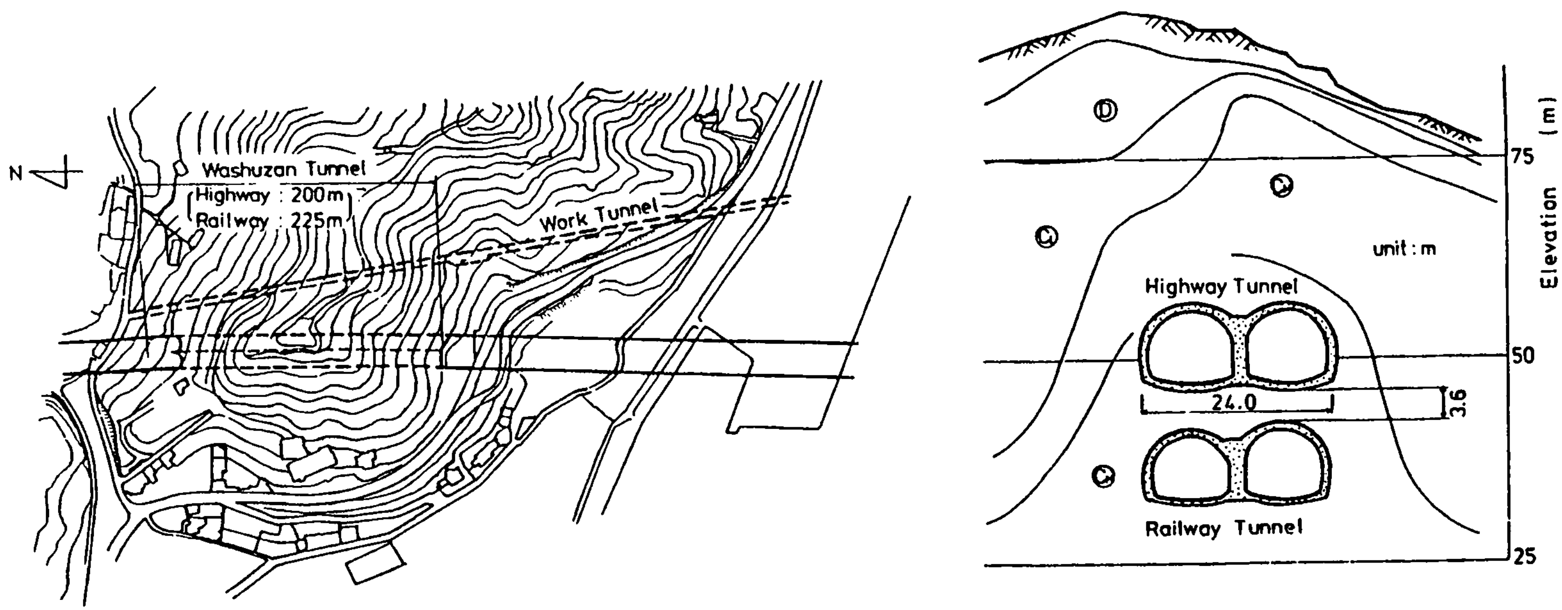


Figure 11. Plane view of tunnel construction site and tunnel cross-section (after Reference 54)

worked out, e.g. on the basis of the weight of the overburden. The elastic modulus of the rock mass E_R can be determined dividing σ_y^o by the corresponding normalized stress (cf. equation 52). The accuracy of this modulus is controlled, introducing it in a finite element simulation of the excavation and comparing the numerical results with the displacements measured in the field.

If the back calculated modulus is sufficiently close to the one adopted during design, it can be concluded that the initial hypotheses on the characteristics of the rock mass are in agreement with the actual *in situ* conditions and no changes in the design are needed. On the contrary, if the back calculated modulus is markedly lower than the design value, some modifications of the original design could be necessary.

The normalized stresses can also be adopted for evaluating a set of equivalent nodal forces that correspond to the right-hand-side term of equation (51) and that lead, through a standard finite element analysis, to the strain state around the opening. Note that, due to definition adopted for these forces, the finite element analysis cannot define the actual stress state.

The excavation is stable if the above shear strain distribution does not exceed the 'critical strain' of the rock. Experimental investigation has proved in fact[57,58] that the critical strain, defined as the ratio between rock strength and elastic modulus, is a parameter suitable for

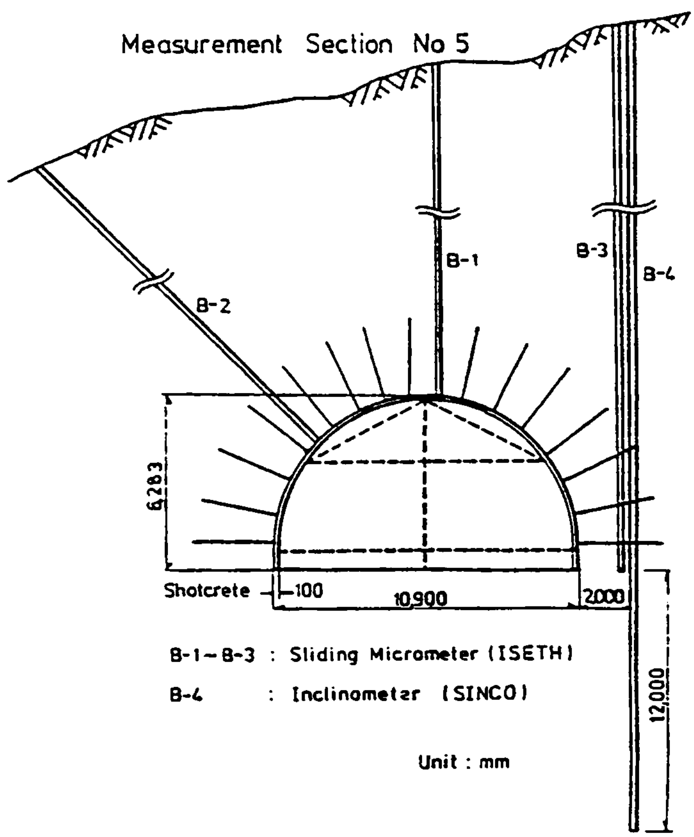


Figure 12. Positions of the boreholes for inclinometers and micrometers ($B-1 \cdots B-4$) and directions of the convergence measurements (dashed lines) (after Reference 54)

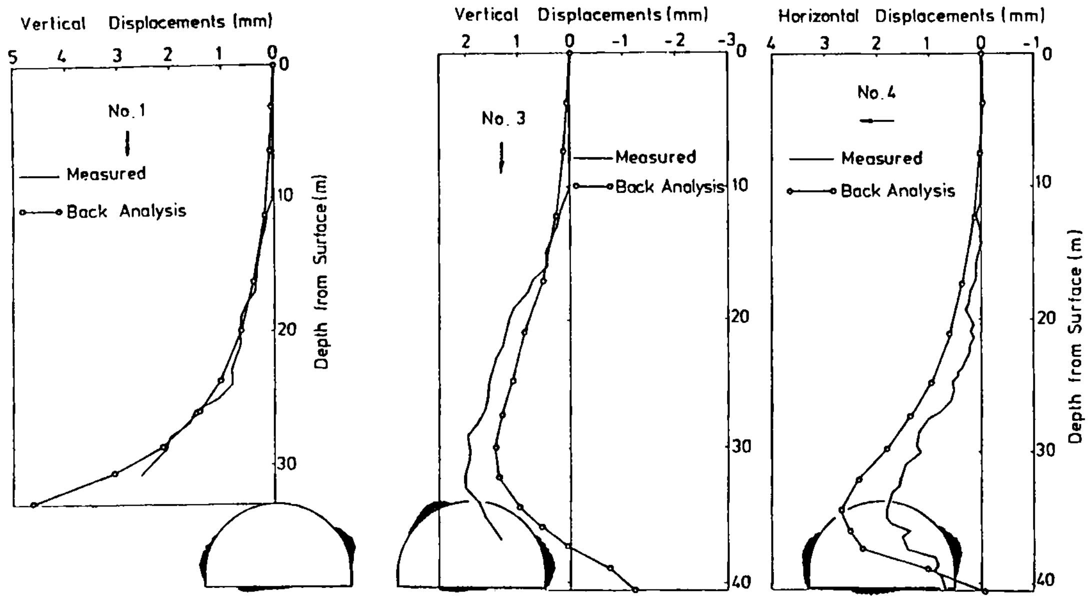


Figure 13. Comparison of back analyzed displacements with measured values (after Reference 54)

defining the limit mechanical resistance of the rock mass. An interesting characteristic of the critical strain is that its value is weakly influenced by existing joints, specimen size and other structural characteristics. Therefore, in addition to costly *in situ* tests, relatively cheap laboratory tests can also be used for determining a reliable value of this parameter. An approximate method has been also presented in Reference 59 for determining, on the basis of the mentioned quantities, the so-called 'plastic' zone that may develop around the opening.

The described technique has been applied to a back analysis problem[54] concerning two double track railway tunnels and two double lane highway tunnels excavated in a relatively homogeneous weathered granite (Figure 11). Although the tunnels have a limited length (about 200 m), this project presents some notable characteristics due to the presence of four parallel tunnels, situated very close to each other, and to the limited overburden (about 30 m).

Displacement measurements were conducted during the construction of a work tunnel, excavated in advance of the main tunnels and almost parallel to them. To this purpose sliding micrometers and inclinometers, installed from the ground surface, were used. The positions of these instruments, and the directions chosen for some convergence measurements, are shown in Figure 12. These displacements were adopted as input data in a back analysis of the rock mass elastic modulus, in which suitable values for the vertical *in situ* stress and Poisson's ratio of the rock were assumed, and the shotcrete lining was neglected because of its small thickness.

In Figure 13 a comparison is presented between some of the measured displacements and the corresponding ones obtained from a finite element analysis based on the back calculated modulus. Finally, the strain distribution at the end of constructions of the main tunnels, determined with a finite element analysis (Figure 14) in which the back calculated normalized initial stresses were introduced, is shown in Figure 15.

Alternative approaches for back analyses related to tunnels have been proposed and applied by various authors. As an example, two of them are mentioned here that can be seen, in a sense, as derived from two different 'philosophies' for the interpretation of field measurements.

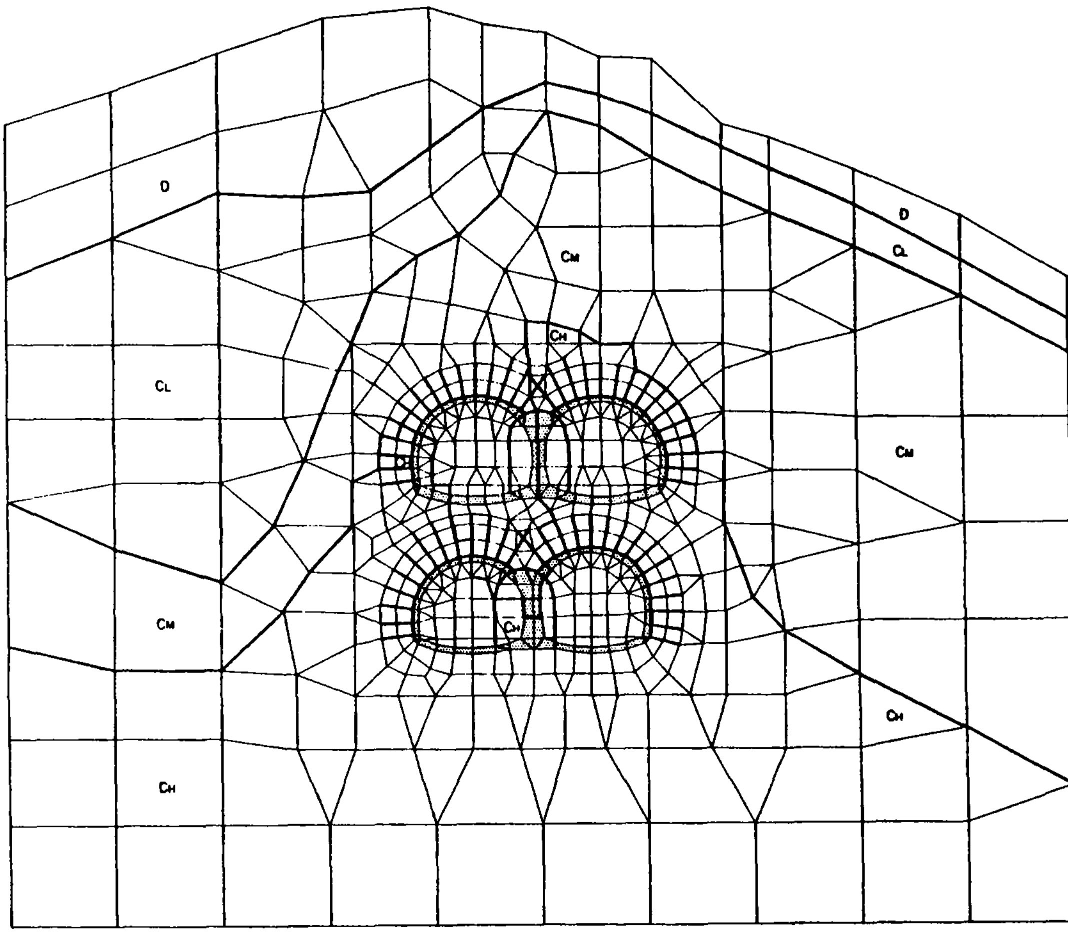


Figure 14. Finite element mesh for the main tunnels (after Reference 54)

The approach proposed in reference 60 is among those that cast the back analysis problem within a relatively sophisticated mathematical framework. In fact, least squares and a maximum likelihood formulations are used for determining the elastic constants of a layered soil deposit surrounding a shallow railroad tunnel. The same authors also discuss a procedure for evaluating the coefficient of lateral earth pressure K_o on the basis of displacements measured during excavation. They show that a unique 'optimum' for this parameter does not exist, but that a range can be determined within which the optimal value of the parameter will fall.

A more practically orientated approach,[61-63] has been adopted for interpreting the measurements performed during the construction of large underground cavities and of powerhouse caverns. The back analyses are based on a sequence of finite element calculations where the input data are suitably modified by the designer until a satisfactory agreement is reached between numerical and experimental results. Here the optimization algorithm is replaced by the judgement and the experience of the engineer. In spite of its simplicity this procedure, when properly applied, is able to provide satisfactory results especially when dealing with complex, three dimensional,

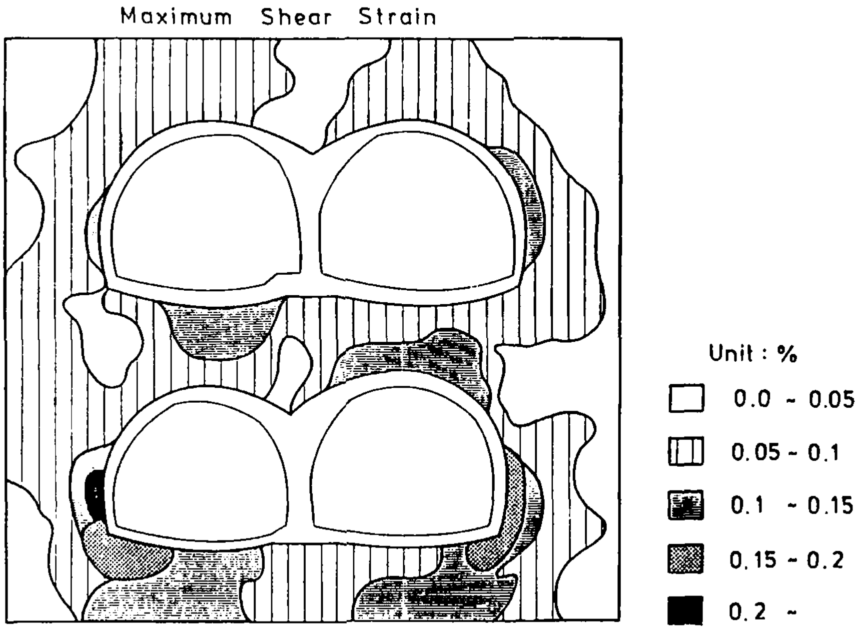


Figure 15. Maximum shear strain distribution around the main tunnels evaluated by back analysis (after Reference 54)

problems that would be perhaps difficult to handle by means of the previously described minimization methods.

CONCLUSION

A survey has been presented of some recent developments of the numerical techniques for back analysis in the field of geomechanics. These methods are seen as practical tools for reducing the uncertainty often affecting the parameters to be used in the design of complex geotechnical works or for refining during time, i.e. when data from field observations become available, the parameters adopted at the preliminary design stage.

Various problems have been discussed concerning the use of *in situ* measurements performed during construction and/or excavation works for identifying, or calibrating, the pressure exerted by the rock on supporting structures, the *in situ* stress state and the material parameters of rock masses. These problems have been considered only in the context of static material response and with particular reference to tunnelling problems.

Nowadays, back analysis techniques are more and more frequently used in practice because of two main reasons: (a) the development of fast, small size computers that permit carrying out the large amount of calculations required by the majority of back analysis techniques in a limited time and with an acceptable cost; (b) the growing use in geotechnical engineering of numerical (computer oriented) methods for stress analysis, such as the finite element method, that represent the basis of any calibration procedure.

A parallel increase can also be observed in the research for new, more efficient back analysis techniques. Considering the present state of the research it appears that, among various possible topics for further studies, the following are of particular interest:

1. Techniques for the calibration of complex constitutive models, for static material response, from

field observations. In fact, most applications presented in the literature concern relatively simple material models, such as elastic or elastic–ideally plastic.

2. Extensions of back analysis techniques to dynamic problems. This is a topic intensively studied in mechanical engineering, but only a limited number of contributions have been presented so far in geotechnical engineering. Such a type of studies could have a notable importance in connection with soil/rock dynamics and seismic problems.
3. Further studies toward the application to design problems of probabilistic back analysis procedures. This is of particular interest in geomechanics due to the intrinsic non-deterministic aspects of many practical situations.
4. Study of procedures for determining not only the 'best' set of parameters for an '*a priori*' chosen material model, but also for defining among various possible models the one providing the best description of the actual behaviour of the rock mass.

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